The quantum dynamics of heterotic vortex strings

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ABSTRACT: We study the quantum dynamics of vortex strings in $\mathcal{N} = 1$ SQCD with $U(N_c)$ gauge group and $N_f = N_c$ quarks. The classical worldsheet of the string has $\mathcal{N} = (0, 2)$ supersymmetry, but this is broken by quantum effects. We show how the pattern of supersymmetry breaking and restoration on the worldsheet captures the quantum dynamics of the underlying 4d theory. We also find qualitative matching of the meson spectrum in 4d and the spectrum on the worldsheet.

Contents

1. Introduction

Vortex strings have proven to be a useful probe of four-dimensional quantum dynamics. By studying the $d = 1 + 1$ theory on the string worldsheet, one may extract information about the strongly coupled phase of the underlying four-dimensional theory. For vortex strings in $\mathcal{N} = 2$ supersymmetric theories, this information includes the exact BPS mass spectrum, the Seiberg-Witten curve, and the dimensions of chiral primary operators at superconformal points $[1-4]$. For earlier, related work see $[5, 6]$; for further work in this direction see $[7-16]$.

The purpose of this short letter is to study the quantum dynamics of vortex strings in $\mathcal{N} = 1$ supersymmetric gauge theories. The classical dynamics of these vortices was recently determined in $[17]$. The strings have $\mathcal{N} = (0, 2)$ supersymmetry on their worldsheet and were dubbed "heterotic vortex strings". In this paper we focus on $\mathrm{U}(N_c)$ SQCD with $N_f = N_c$ fundamental flavors. The low-energy quantum physics of this theory was understood some years ago by Seiberg in terms of a quantum deformation of the moduli space of vacua $[18]$. We will show that this four-dimensional behavior can be reproduced by studying the $(0, 2)$ worldsheet dynamics of the vortex string. The relevant physics on the worldsheet is dynamical supersymmetry breaking and supersymmetry restoration. We will also find qualitative matching of the meson spectrum in 4d and the spectrum of the worldsheet.

Before jumping into the details, let us pause to review the basic philosophy. When studying the quantum dynamics of solitons there are typically two different scales in the game: Λ, the scale of strongly coupled quantum effects and M, the scale of the symmetry breaking that supports the soliton. The approach one must take to study soliton dynamics depends on the ratio of these two scales. When $\Lambda \gg M$, one must first deal with the

quantum dynamics in four dimensions and subsequently look for solitons in the low-energy effective theory. In contrast, when $M \gg \Lambda$, one should instead look for solitons in the microscopic four-dimensional theory and then subject these solutions to a traditional semi-classical quantization. Although these approaches are valid in different regimes, comparing the results can sometimes be profitable, especially if certain details do not depend on the ratio Λ/M. For example, the equivalence of Liouville theory and the two-dimensional black hole [19] was understood in this way through the study of domain walls [20]. In the following pages we will examine the quantum dynamics of vortex strings using these two different approaches. The punchline is that the two routes commute, and the semi-classical quantization of the $(0,2)$ worldsheet theory mirrors the quantum dynamics of the four-dimensional theory. Stated another way, one can recover the 4d quantum dynamics by studying the 2d dynamics of the string.

2. SQCD: $N_f = N_c$

Our focus in this paper is $\mathcal{N} = 1$ SQCD, with $\mathrm{SU}(N_c)$ gauge group and $N_f = N_c \equiv N$ fundamental flavors. Each flavor consists of two chiral superfields, Q and $\tilde{Q}$, transforming in the $\mathbf{N}$ and $\bar{\mathbf{N}}$ representations of the gauge group respectively. The classical theory has global symmetry group

$$\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R \tag{2.1}$$

under which the chiral superfields Q transform as $(\mathbf{N}, \mathbf{1}, 1, 1/N, 0)$ while $\tilde{Q}$ transform as $(\mathbf{1}, \bar{\mathbf{N}}, 1, -1/N, 0)$. Note that the charge assignment refers to the scalar component of Q and $\tilde{Q}$; the fermionic components do transform under the $\mathrm{U}(1)_R$ R-symmetry, as does the gaugino. The R-symmetry is non-anomalous. However $\mathrm{U}(1)_A$ does suffer from an anomaly, leaving a remnant $\mathbf{Z}_{2N}$.

The low-energy physics of this theory is well understood [18]. The classical theory has a moduli space of vacua, given by solutions to the D-flatness conditions. These solutions are parameterized in a gauge invariant fashion by the meson chiral superfield,

$$M_{ij} = \tilde{Q}_i Q_j \tag{2.2}$$

together with a pair of baryon chiral superfields,

$$B = \epsilon_{a_1 \ldots a_{N_c}} Q_1^{a_1} \ldots Q_{N_c}^{a_{N_c}} \quad , \quad \tilde{B} = \epsilon_{a_1 \ldots a_{N_c}} \tilde{Q}_1^{a_1} \ldots \tilde{Q}_{N_c}^{a_{N_c}} \tag{2.3}$$

These are not independent. They obey the classical constraint

$$\det M - B\tilde{B} = 0 \tag{2.4}$$

The light fields in the classical theory are M, B and $\tilde{B}$ subject to the constraint (2.4). The resulting manifold has singularities at $\tilde{B} = B = 0$ and $\mathrm{rank}(M) < N - 2$. These singularities reflect the existence of new massless gluons which emerge when the symmetry breaking is less than maximal.

The situation in the quantum theory is different. The classical constraint (2.4) is corrected [18],

$$\det M - B\tilde{B} = \Lambda_{4d}^{2N} \tag{2.5}$$

where $\Lambda_{4d}^{2N} = \mu^{2N} e^{-4\pi^2/e^2(\mu)+i\theta}$ is proportional to the one-instanton action. The manifold defined by (2.5) is smooth. The singularities of the classical moduli space have been resolved, reflecting the confining nature of the quantum theory

2.1 Vortices in the low-energy theory

Let us now describe how vortices appear in the low-energy theory. The $\mathrm{SU}(N_c)$ theory does not have the topology to support vortex strings. To introduce vortices we deform the theory by gauging the $\mathrm{U}(1)_B$ baryon symmetry. Of the low-energy fields, M is neutral under $\mathrm{U}(1)_B$ while B and $\tilde{B}$ have charge $+1$ and -1 respectively. We introduce a Fayet-Iliopoulos (FI) parameter $v^2 > 0$ for $\mathrm{U}(1)_B$ which imposes the D-flatness condition on the scalar fields,

$$|B|^2 - |\tilde{B}|^2 = v^2 \tag{2.6}$$

Since $v^2 > 0$, we necessarily have $B \neq 0$ in vacuum. The $\mathrm{U}(1)_B$ invariant combination $\tilde{B}B$ is then determined by the meson vev through the constraint (2.4) or (2.5).

The presence of the FI parameter ensures that the $\mathrm{U}(1)_B$ gauge symmetry is broken which, in turn, guarantees the existence of vortex strings. One may still trust the low-energy effective theory (2.5) provided one hits the 4d strong coupling scale Λ_{4d} before the Higgs mechanism induced by v^2 kicks in. This means $\Lambda_{4d} \gg ev$.

The question that will concern us in this paper is: When are the vortex strings BPS? The equations describing a BPS string are the first order vortex equations,

$$F_{12}^B = e^2(|B|^2 - |\tilde{B}|^2 - v^2) \ , \quad \mathcal{D}_1 B = i\mathcal{D}_2 B \ , \quad \mathcal{D}_1 \tilde{B} = i\mathcal{D}_2 \tilde{B} \tag{2.7}$$

The key observation, which permeates this paper, is that these equations have solutions only when $\tilde{B} = 0$ [21–23]. This fact follows from a standard theorem in mathematics which states that there exists no non-zero holomorphic section of negative degree — see, for example, [24]. To repeat: BPS vortices exist only if $\tilde{B} = 0$. When $\tilde{B} \neq 0$, any vortex solution must necessarily be non-BPS.

Let us examine the conditions for BPS vortices in both the classical and quantum theories. In the classical theory, the constraint (2.4) allows us to happily sit in the vacuum $|B|^2 = v^2$, with $\tilde{B} = M = 0$. This vacuum is identified by the surviving $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ global symmetry. It enjoys BPS vortices. Indeed, BPS vortices continue to survive in any vacuum with $\mathrm{rank}(M) < N$. However, once $\mathrm{rank}(M) = N$, the constraint (2.4) requires $\tilde{B} \neq 0$ and the vortex is no longer BPS.

Things are rather different once we take into account the 4d quantum dynamics. In the vacuum $M = 0$, with $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ global symmetry, we must necessarily have $\tilde{B} \neq 0$ and the vortices break supersymmetry. To find a BPS vortex, we must now sit in a vacuum with $\det M = \Lambda^{2N}$. We learn that, as far as the vortex strings are concerned, the

4d quantum dynamics both giveth and taketh away. Quantum effects induce both supersymmetry breaking, in which classically BPS vortices no longer preserve supersymmetry, as well as supersymmetry restoration, in which classically non-BPS vortices are rendered BPS in the full quantum theory.

3. The microscopic vortex theory

We now return to the original $SU(N)$ gauge theory. We again gauge $U(1)_B$ and introduce a FI parameter v^2. This induces vevs for the squarks that break the $U(N)$ gauge group completely. This time we work in the regime $ev \gg \Lambda_{4d}$, ensuring that gauge group is spontaneously broken before strong coupling effects take hold. In this regime, we may treat the vortex string in the $U(N)$ theory in a semi-classical manner. We will find that by studying dynamical supersymmetry breaking on the string worldsheet we will be able to reproduce the quantum deformation (2.5).

We start by discussing the string dynamics in the classical vacuum $|B|^2 = v^2$ and $M = \tilde{B} = 0$. In this vacuum the classical vortex is BPS, preserving one half of the four supercharges. The microscopic theory of the vortex string in this vacuum was determined in [17]. It is given by the $\mathcal{N} = (0,2)$ supersymmetric $\mathbf{C} \times \mathbf{CP}^{N-1}$ sigma-model. The factor $\mathbf{C}$ describes the translational and Goldstino modes of the string. The $\mathbf{CP}^{N-1}$ part describes the internal orientation modes of the vortex in color and flavor space [7, 8]. The theory is most simply described in terms of a gauged linear sigma-model for N homogeneous coordinates ϕ_i

$$\mathcal{L}_{\text{vortex}} = |\mathcal{D}_m \phi_i|^2 + 2i\left(\bar{\xi}_{-i}\mathcal{D}_+ \xi_{-i} + \bar{\xi}_{+i}\mathcal{D}_- \xi_{+i}\right) + \frac{\theta}{2\pi}\epsilon_{\mu\nu}\partial^\mu A^\nu$$
$$- D(|\phi_i|^2 - r) - \left(\bar{\zeta}_-\bar{\xi}_{+i}\phi_i + \bar{\phi}_i\xi_{+i}\zeta_-\right) \tag{3.1}$$

The Lagrange multiplier D imposes the constraint $\sum_i |\phi_i|^2 = r$. Dividing by the $U(1)$ gauge action $\phi_i \to e^{i\alpha}\phi_i$ reduces the theory to a sigma-model with $\mathbf{CP}^{N-1}$ target space of size [7, 11]

$$r = \frac{2\pi}{e^2} \tag{3.2}$$

The θ-angle on the worldsheet is inherited from the θ-angle in the four-dimensional bulk [2, 11]. Note that the Grassmannian Lagrange multiplier ζ_- restricts the right-moving fermions to lie in the tangent space: $\phi_i\bar{\xi}_{+i} = 0$. In contrast the left-moving fermions ξ_{-i} are unconstrained[1]. The $(0,2)$ $\mathbf{CP}^{N-1}$ sigma-model can be thought of as a deformation of the more familiar $\mathcal{N} = (2,2)$ $\mathbf{CP}^{N-1}$ sigma-model [24–26] in which the auxiliary "σ-field" is removed, together with an auxiliary fermion. In particular, the $(0,2)$ $\mathbf{CP}^{N-1}$ sigma-model (3.1) has no four-fermi interaction.

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[1]There is a subtlety here. The left-moving fermi zero modes ξ_{-i} are non-normalizable due to an infrared divergence. Nonetheless, they are needed to avoid a sigma-model anomaly on the worldsheet. These problems can be circumvented by considering the four-dimensional theory a torus of area $A \gg 1/e^2v^2, 1/\Lambda^2$.

The worldsheet Lagrangian (3.1) is invariant under the global symmetry group,

$$SU(N)_L \times SU(N)_R \times U(1)_A \times U(1)_R \qquad (3.3)$$

inherited from the subgroup of the 4d symmetry (2.1) preserved in the vacuum. (The worldsheet $SU(N)_L$ actually descends from the unbroken diagonal combination of the 4d $SU(N)_L$ and the gauge group $SU(N)$). The ϕ_i transform as $(\mathbf{N}, \mathbf{1}, +1, 0)$; the right-handed fermions ξ_{+i} transform as $(\mathbf{N}, \mathbf{1}, +1, -1)$, while the left-handed fermions ξ_{-i} transform as $(\mathbf{1}, \mathbf{N}, -1, -1)$. The $U(1)_R$ symmetry is non-anomalous, while the $U(1)_A$ symmetry suffers an anomaly, leaving a $\mathbf{Z}_{2N}$ remnant. This mimics the behavior of the 4d symmetry. The agreement between the anomaly structure in 4d and on the worldsheet is no coincidence. In both cases, the $U(1)_A$ anomaly can be understood by examining the fermi zero modes carried by the relevant instanton. Yet, the instanton in the 2d and 4d theories are different descriptions of the same object: the worldsheet instanton can be thought of as the Yang-Mills instanton, trapped to lie within the vortex string [2]. As one sends $v^2 \to 0$, the vortex string melts away but the worldsheet instanton remains, changing smoothly into the Yang-Mills instanton. The number of fermi zero modes is dictated by an index and remains unchanged in this limit, ensuring that the anomalies in the bulk and on the worldsheet agree.

3.0.1 Dynamical supersymmetry breaking

We now turn to the quantum dynamics of the microscopic worldsheet theory. We will show that the Lagrangian (3.1) dynamically generates an expectation value for the Lagrange mutliplier D. Since D is the auxiliary field in an $\mathcal{N} = (0, 2)$ vector multiplet, and may be written as a supersymmetry variation $\delta\zeta_- = \epsilon_- D$, this means the theory dynamically breaks supersymmetry. In fact, the expectation value for D was already shown 30 years ago as a simple application of the $1/N$ expansion in the bosonic sigma-model [27, 25], and it is not hard to check that the presence of fermions in the $(0, 2)$ model do not change this conclusion[2]. Ignoring the fermions, one first integrates out the bosonic fields ϕ_i to leave the partition function

$$Z_{\text{bose}} = \int dD \, dA_\mu \, \exp\left(-N \operatorname{Tr} \log\left[-(\partial_\mu + iA_\mu)^2 - D\right] + i\int d^2x \, Dr + \frac{\theta}{2\pi}\epsilon_{\mu\nu}\partial^\mu A^\nu\right)$$

The Lorentz invariant ground state has $A_\mu = 0$, with D sitting at the stationary point,

$$ir + N\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - D + i\epsilon} = 0 \qquad (3.4)$$

The integral can be performed exactly to give the supersymmetry breaking expectation value,

$$D = 4\Lambda_{2d}^2 \qquad (3.5)$$

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[2]In contrast, supersymmetry is not broken in the $\mathcal{N} = (2, 2)$ supersymmetric model. This can be traced to the presence of an extra bosonic field σ which bears the burden of the expectation value instead of D.

where the strong coupling scale on the worldsheet is given at one-loop by $\Lambda_{2d}^N = \mu^N e^{-2\pi r + i\theta}$. Recalling the relationship (3.2), we can match the Λ parameters on the worldsheet and the bulk by first running e^2 down to the scale of the vortex tension, and subsequently running r down to strong coupling: $\Lambda_{4d}^{2N} = v^N \Lambda_{2d}^N$. The dynamical supersymmetry breaking on the worldsheet mirrors the behavior we saw in studying the four-dimensional quantum dynamics: the classical vortex is BPS, but is rendered non-BPS by quantum effects. We will shortly see what becomes of the vortex in the other 4d vacua.

3.0.2 The spectrum

One of the most striking feature of vortex strings in $\mathcal{N} = 2$ theories is that the BPS mass spectrum of the worldsheet coincides with the BPS spectrum of the bulk theory in the phase with $v^2 = 0$ [5, 6, 1, 2]. What is the story in our $\mathcal{N} = 1$ theory? There are no longer BPS particle states with their associated non-renormalization theorems, and exact agreement of the spectrum would be optimistic. Nonetheless, one might hope to match qualitative features of the spectrum such as quantum numbers. Most notably, one would hope to see the confining nature of the $\mathcal{N} = 1$ theory from the perspective of the worldsheet.

Why do we expect even qualitative agreement between the spectrum of the worldsheet and the spectrum of the 4d theory in the confining phase? Let us return momentarily to the situation with $ev \ll \Lambda_{4d}$. At distance scales $\ll 1/ev$, the 4d particle spectrum is largely unaffected by the Higgs mechanism. Some of these particles are likely to form weakly bound states with the vortex string. As we now increase ev/Λ_{4d}, those bound states which remain light ($\ll ev$) will appear as internal excitations of the microscopic worldsheet theory.

The spectrum of the 4d theory contains both baryons and mesons. Once we turn on $v^2 \neq 0$, the baryons are all screened in the bulk and unlikely to form bound states with the vortex. This leaves us with the spectrum of mesons, consisting of the massless fields $M_{ij} = \tilde{Q}_i Q_j$ transforming in the bi-fundamental representation of the $SU(N)_L \times SU(N)_R$ global symmetry group, together with massive fields $Q_i^\dagger Q_j$ and $\tilde{Q}_i \tilde{Q}_j^\dagger$ transforming in the singlet and adjoint representations. Because of confinement, there are no physical states transforming in the fundamental representations of the global symmetry group. How much of this is seen by the string?

Let us firstly recall some basic facts about $\mathbf{CP}^{N-1}$ sigma-models. The purely bosonic $\mathbf{CP}^{N-1}$ model is well known to exhibit confinement [27, 25]. This was previously noted in the context of the vortex theory in [11]. Confinement occurs because the auxiliary $U(1)$ gauge field in (3.1) becoming dynamical after integrating out ϕ_i and the Coulomb potential grows linearly in two-dimensions. This means that the bosonic model contains no physical states transforming in the $\mathbf{N}$ or $\bar{\mathbf{N}}$ representations of the global $SU(N)$ symmetry group. All physical states live in either singlet or adjoint representations.

Typically the physics of the $\mathbf{CP}^{N-1}$ model changes drastically in the presence of fermions and often the model no longer confines [25]. The mechanism by which this happens is rather interesting. As we have seen above, if we couple N fermions, each charged under the worldsheet $U(1)$ gauge field, then the $U(1)_A$ axial symmetry is anomalous, broken to the discrete subgroup $\mathbf{Z}_{2N}$. Moreover, under the right conditions — a caveat that

will be clarified below — strong coupling effects induce a condensate of fermion bilinears $\bar{\xi}_i \xi_i = \bar{\xi}_{+i} \xi_{-i} - \bar{\xi}_{-i} \xi_{+i}$,

$$\langle \bar{\xi}_i \xi_i \rangle \sim \Lambda_{2d} \tag{3.6}$$

This condensate spontaneously breaks the anomaly free discrete symmetry $\mathbf{Z}_{2N} \to \mathbf{Z}_2$. This ensures the existence of N isolated vacua of the model, together with the associated particle states arising from kinks which interpolate between the different vacua. These kinks transform in the fundamental representation of $\mathrm{SU}(N)$ and, in fact, can be identified with the original degrees of freedom ϕ_i [25]. The condensate (3.6) occurs in the $\mathcal{N} = (2, 2)$ $\mathbf{CP}^{N-1}$ model where it is a key element in matching the spectrum of the vortex worldsheet to the spectrum of $\mathcal{N} = 2$ four-dimensional theories. In that case, neither the worldsheet theory, nor the 4d theory, confines.

What of our $\mathcal{N} = (0, 2)$ $\mathbf{CP}^{N-1}$ model? The worldsheet theory includes fermions, so we may expect the mechanism above to again be at play. However, the four-dimensional physics is certainly confining. What's going on? In fact, the condensate (3.6) cannot form in the $(0, 2)$ $\mathbf{CP}^{N-1}$ sigma-model. The reason is that, in contrast to the $(2, 2)$ model, the theory has a chiral $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ flavor symmetry which would be broken by (3.6). However, there can be no spontaneous symmetry breaking in two dimensions: the ground state wavefunction spreads to preserve all global symmetries in the vacuum [28, 29]. The preservation of the continuous $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ acts as a guardian of the discrete Z_{2N} symmetry and neither are broken in the vacuum. This ensures that the worldsheet theory confines, and all particle states transform in the adjoint, singlet or bi-fundamental representations of $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$, in broad agreement with the 4d meson spectrum.

More is known about the worldsheet theory. It can be shown that at at least $N^2 - 1$ of the states are massless. This follows by treating the large N theory as QED_2 with N massless flavors [30 – 32] to which is almost reduces. (The presence of the constraint for the right-moving fermions means it differs slightly, but this does not change the conclusions here). It also follows on general grounds from the $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ chiral symmetry [33]. These massless modes are not Goldstone bosons but rather are related to the fact that the theory lies in the Berezinski-Kosterlitz-Thouless phase, with power-law decay of the correlator [34, 35]

$$\langle \bar{\xi}_i(x) \xi_i(x) \, \bar{\xi}_i(0) \xi_i(0) \rangle \sim x^{-2/N-2} \tag{3.7}$$

It is natural to identify these massless modes as threshold bound states of the chiral mesons M_{ij} with the vortex strings. The remaining particles on the worldsheet have mass of order Λ_{2d} and presumably originate from bound states of the massive 4d mesons.

3.1 Exploring the Higgs branch

We have studied the vortex dynamics in the vacuum $M = 0$, where quantum effects dynamically break supersymmetry on the vortex worldsheet. Let us now place ourselves in other vacua of the 4d theory. We first need to understand how the classical vortex dynamics responds to changing the meson vev M. The answer to this question in $\mathcal{N} = 2$

theories was given in [3] and a similar analysis applies here. The key physics involved is similar to that described above. Turning on an expectation value for M in the classical 4d theory necessarily means turning on a vev for $\tilde{Q}$. But, by the same argument given after equation (2.7), there can be no solution to the non-Abelian first order vortex equations if $\tilde{Q}$ is sourced and carries a non-trivial profile. In the non-Abelian theory, the vortex can avoid this by sitting in part of the gauge group unaffected by the vev for $\tilde{Q}$. This removes some of the moduli ϕ_i. It is simple to adapt the argument given in [3] to the present case. One finds that BPS vortices are given by the zeroes of the worldsheet potential,

$$\delta\mathcal{L}_{\text{vortex}} = \bar{\phi}_i \, \frac{M_{ij}^\dagger M_{jk}}{v^2} \, \phi_k + \bar{\xi}_{i-} \frac{M_{ij}}{v} \, \xi_{j+} + \text{h.c.} \tag{3.8}$$

The overall constant in front of this potential is undetermined, while the factors of v here are required on dimensional grounds. It is simple to show that this deformation preserves $\mathcal{N} = (0,2)$ supersymmetry. (For example, it may be written in superspace language as a $(0,2)$ superpotential). There exist zeroes of the potential in (3.8), subject to the D-term constraint $|\phi_i|^2 = r$ only when $\text{rank}(M) < N$. This is indeed the condition for the existence of classical BPS vortices.

Let us now see how this conclusion is changed by quantum dynamics on the worldsheet. We again work in the $1/N$ expansion and integrate out ϕ_i. As before, the smoking gun for broken supersymmetry is the expectation value of D. Including now the extra term (3.8), we find that equation (3.5) is replaced by,

$$\det\left(\frac{M^\dagger M}{v^2} + D\,1_N\right) = 4^N \Lambda_{2d}^{2N} \tag{3.9}$$

To leading order in $1/N$, D has vanishing expectation value, and supersymmetry is restored, providing that we tune the the meson expectation value to

$$\det M \sim v^N \Lambda_{2d}^N \sim \Lambda_{4d}^N \tag{3.10}$$

where $\sim$ refers to the fact that we have not fixed the overall constant. This requirement for BPS vortices reproduces the analysis of vortices in the quantum deformed moduli space (2.5). It is noteworthy that when $\text{rank}(M) = N$, the classical worldsheet theory spontaneously breaks supersymmetry, yet quantum effects may restore supersymmetry in the infra-red: the classical non-BPS vortex is rendered BPS.

4. Soft breaking from $\mathcal{N} = 2$

There is another limit in which we can profitably understand the dynamical supersymmetry breaking on the worldsheet of the vortex string, and that is in softly broken $\mathcal{N} = 2$ theory. We introduce a new scale μ, which gives a mass to the adjoint scalar Φ in the $\mathcal{N} = 2$ vector multiplet. The 4d superpotential is given by

$$\mathcal{W}_{4d} = \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_i (\Phi - m_i) Q_i + \mu \Phi^2 \tag{4.1}$$

where we have also introduced complex masses m_i for each flavor.

The 4d theory becomes strongly coupled at the scale $\Lambda_{N=2} = \mu e^{-4\pi^2/e^2 N + i\theta/N}$ which, one can check, is equal to Λ_{2d}. When $\Lambda_{N=2} \gg ev$, we should first study the strongly coupled dynamics in four-dimensions. Our goal is to understand the expectation value of $\tilde{B}$ as a function of $\Lambda_{N=2}$ and the masses m_i, for this will tell us when BPS vortices exist. We work in several steps. Let us first consider the $\mathcal{N} = 2$ theory with $\mu = 0$. The theory has a Coulomb branch, parameterized by the vev of Φ. However, since we ultimately wish to gauge $U(1)_B$ and turn on a FI parameter v^2, we are not interested in the theory at an arbitrary point on the Coulomb branch but rather in the special vacuum which lies at the root of the baryonic Higgs branch. Classically this is given by $\Phi = \mathrm{diag}(m_1, \ldots, m_N)$ where N quarks are massless and may condense once $v^2 \neq 0$. Quantum mechanically, the information about the dynamics is encoded in the Seiberg-Witten curve, a rational curve in $\mathbf{C}^2$ whose periods determine the couplings of low-energy fields. At the root of the baryonic Higgs branch, this curve degenerates, reflecting the presence of the massless quarks. It is given by,

$$F(v, t) = [\Lambda_{N=2}^N t - \prod_{i=1}^N (v - m_i)^2][t - 1] \equiv C_L(t, v) C_R(t) = 0 \tag{4.2}$$

The next step is to introduce μ, softly breaking the theory to $\mathcal{N} = 1$. The low-energy dynamics is now captured by a curve in $\mathbf{C}^3$ instead of $\mathbf{C}^2$. (This is particularly apparent in the five-brane picture where the curve is the brane, and soft-breaking corresponds to rotating the brane in transverse directions). For the degenerate curve (4.2), the effect of the soft breaking is simple: the curve once again factorizes and is given by [36, 37]

$$
\begin{aligned}
C_L : \quad & \Lambda^N t = \prod_{i=1}^N (v - m_i) \ , \quad \omega = 0 \\
C_R : \quad & t = 1 \ , \quad \omega = 2\mu v
\end{aligned}
\tag{4.3}
$$

In [36, 37], the expectation of the baryonic operator $\tilde{B}B$ was argued to be the distance between these two branches at $\omega = 0$. This gives us

$$\tilde{B}B = \Lambda^N - \prod_{i=1}^N (-m_i) \tag{4.4}$$

The final step is to gauge $U(1)_B$ and turn on the FI parameter v^2. Once again, the vortices are only BPS if $\tilde{B} = 0$. The condition for the existence of BPS vortices is therefore

$$\Lambda^N = \prod_{i=1}^N (-m_i) \tag{4.5}$$

Let us now see how to reproduce this result from the quantum dynamics of the worldsheet. When $\Lambda_{N=2} \ll ev$, we should treat the string semi-classically. In the limit $\mu = 0$, where we have 4d $\mathcal{N} = 2$ supersymmetry, the low-energy dynamics of the string is governed by the $\mathcal{N} = (2, 2)$ $\mathbf{C} \times \mathbf{CP}^{N-1}$ sigma-model [7, 8], with a potential induced by the masses m_i

for the 4d hypermultiplets [9, 1, 2]. The worldsheet action (3.1) for the $\mathbf{CP}^N$ part is to be augmented with a scalar σ and a fermion ζ_+ and the additional terms

$$\delta\mathcal{L}_{(2,2)\text{ vortex}} = \sum_{i=1}^{N} |\sigma - m_i|^2 |\phi_i|^2 - \bar{\zeta}_+ \phi_i \bar{\xi}_{-i} + \bar{\xi}_{-i}(\sigma - m_i)\xi_{+i} + \text{h.c.} \qquad (4.6)$$

As in previous sections, we may integrating out the chiral multiplets containing ϕ_i. This now yields an effective twisted superpotential for the twisted chiral multiplet Σ whose lowest component is the scalar σ [26, 24, 38]

$$\mathcal{W}(\Sigma) = -\frac{1}{2\pi} \sum_{i=1}^{N} (\Sigma - m_i) \log\left[\left(\frac{\Sigma - m_i}{M_{UV}}\right) - 1\right] - (r + i\theta/2\pi)\Sigma \qquad (4.7)$$

with M_{UV} the renormalization subtraction point. The scalar potential energy is given by $(K_{\sigma\bar{\sigma}})^{-1}|\partial\mathcal{W}/\partial\sigma|^2$ where K is the (unknown) Kähler potential. Assuming K is smooth, the N vacuum states lie at

$$\frac{\partial\mathcal{W}}{\partial\sigma} = 0 \quad \Rightarrow \quad \prod_{i=1}^{N} (\sigma - m_i) = \Lambda_{2d}^N \qquad (4.8)$$

When the 4d theory is softly broken to $\mathcal{N} = 1$ by the addition of the mass μ, the worldsheet theory is softly broken to $\mathcal{N} = (0,2)$ in way that was determined in [17]. When $\mu \ll \Lambda_{2d}$, it is appropriate to view the breaking within the effective worldsheet theory determined by (4.7). The $\mathcal{N} = (0,2)$ deformation mixes a would-be Goldstino mode from the $\mathbf{C}$ part of the $\mathcal{N} = (2,2)$ theory with the $\mathbf{CP}^{N-1}$ fields. The net effect on the worldsheet is to induce a mass for σ [17], so that the scalar potential now reads

$$V = (K_{\sigma\bar{\sigma}})^{-1}\left|\frac{\partial\mathcal{W}}{\partial\sigma}\right|^2 + |\mu\sigma|^2 \qquad (4.9)$$

We once again see that the worldsheet dynamics mirrors the 4d quantum dynamics. For generic values of the masses m_i, the vacuum has non-vanishing energy and supersymmetry on the worldsheet is spontaneously broken. However, when the masses are tuned to satisfy (4.5), the solution to (4.8) is given by $\sigma = 0$, and a vacuum with zero energy is recovered. This is in full agreement with the analysis of baryon expectation values in four-dimensions.

Acknowledgments

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